

Temporal fractal structures: Origin of power-laws in the world-wide Web

Bosiljka Tadić*

Jožef Stefan Institute, P.O. Box 3000, 1001 Ljubljana, Slovenia

Using numerical simulations and scaling theory we study the dynamics of the world-wide Web from the growth rules recently proposed in Ref. [1] with appropriate parameters. We demonstrate that the emergence of power-law behavior of the out- and in-degree distributions in the Web involves occurrence of temporal fractal structures, that are manifested in the scale-free growth of the local connectivity and in first-return time statistics. We also show how the scale-free behavior occurs in the statistics of random walks on the Web, where the walkers use information on the local graph connectivity.

I. INTRODUCTION

Complex evolving networks differ from static random graphs in that their size increases in time, thus impacting the linking process in a nontrivial manner. Hence the *emergent* structure of links is related to salient features of the growth processes in the network, which is governed by the individual linking rules [2,3]. In the case of the scale-free structures emerging at large evolution times in networks with *preferential* linking, the universality classes characterized by the same set of scaling exponents can be distinguished, that are based on several *relevant details* of the microscopic linking properties. The theoretical background of the universality classes of dynamic networks is still missing. By study of many particular networks it has been recognized that certain dynamic constraints on the linking processes can change the emergent scale-free behavior [1–3].

The world-wide Web differs from the generic scale-free evolving networks in two ways: (1) it is represented by a *directed graph* and, (2) most importantly, it has a *variable wiring diagram*. Frequent updates of the out-going links, that are peculiar for conduct of the agents in the real Web, makes the wiring diagram of the Web graph changing at the same pace as the graph grows. Whereas, wiring diagram of some other networks changes on much slower scale or not at all [1]. The intimate relationship between structural and growth properties leads to a specific architecture of links in the Web. Recent measurements in the real Web have shown [4] that both out- and in-degree distributions are power-law with *different* exponents, as well as the size of the connected clusters out of the giant component.

Occurrence of power-laws is a remarkable feature in large number of complex evolving networks, that indicates presence of underlying self-organization while the network grows. The emergent hierarchical organization of node ranks is highly relevant for the stability of the network under attacks [5], and for the character of other dynamic processes *on* the network, such as random walk processes [6,7]. Therefore understanding the mechanisms of self-organization that lead to power-laws in the world-wide Web is crucial both for its functional stability and for designing efficient search algorithms [8] and transport processes [9] on the Web graph.

As a step towards realistic modeling of the dynamics of world-wide Web we proposed recently the model [1] which takes into account the basic relevant features of the Web growth: directed linking, rewiring of preexisting links, and bias activity of agents and bias attachment of links. It was shown [1] that, when the degree of rewiring in the graph is adjusted to $\beta \approx 3$ (i.e., to each new added link in the graph there are in the average three updated links among preexisting nodes), the model reproduces fairly well the emergent power-law distributions of the out- and in-degree, and the scaling exponent of the connected components. In the present work we use this model to study the details of the growth process that precedes the emergence of link structure in the Web graph. Particularly, we demonstrate that a spatio-temporal *fractal structure of linking activity* occurs on the growth time scale by successive addition of nodes and the average increase of the number of links by $M = \beta + 1$. The fractal properties of the structure are measured by scaling behavior of the distribution of time intervals of the *first-return* activity to a given node. We show that this activity pattern results in the algebraic increase of the average *local connectivity* $\langle q_\kappa(s, t) \rangle$ at node s with time (κ refers to “out” and “in” links), implying scaling behavior in the underlying local probability distribution. In view of the scaling theory the local connectivity is then related to the emergent degree distribution in the limit of large evolution times [10] $t \rightarrow \infty$. We demonstrate these steps by directly simulating the appropriate quantities, both for out- and in-links. In addition, we show that dynamic processes, such as random walks on the Web graph [6,7], that use the information on the local connectivity may also result in the power-law distributions. Here we compute (for the same graph parameters) the distributions of distances on node hierarchy made by an ensemble of random walkers which utilize parts of locally available information on the Web structure.

II. LINKING RULES AND FRACTAL GROWTH PATTERNS

The growth model is defined by the dynamic rules [1] that can be summarized as follows: At each time step $t > M_0 \geq M$ add a node $i = t$ and create M links. A link is first attempted from the new added node with probability α to a target node k that is selected with the probability $p_{in}(k, t)$, specified below. Else, a link is created between a pair of preexisting nodes (updated link) as follows: A link from node $n < i$ to target node $k < i$ at time $t = i$ occurs with the probability

$$C(n, k, t) = (1 - \alpha) p_{out}(n, t) \times p_{in}(k, t) , \quad (1)$$

where both probability to select an origin of the link $p_{out}(n, t)$ and to select a target node $p_{in}(k, t)$ depend on the current connectivity of these nodes $q_{out}(n, t)$ and $q_{in}(k, t)$, respectively,

$$p_{out}(n, t) = \frac{\alpha + q_{out}(n, t)/M}{(1 + \alpha)t} , \quad p_{in}(k, t) = \frac{\alpha + q_{in}(k, t)/M}{(1 + \alpha)t} . \quad (2)$$

At the moment of addition $q_{out}(i, i) = q_{in}(i, i) = 0$ and increasing in time. Therefore, a ratio of the number of added and updated links $\beta \equiv (1 - \alpha)/\alpha$, which is independent on the actual number of links M , is the control parameter in the model. For simplicity we keep M fixed, assuming that the number of links fluctuate in time around the average value M . Motivation for the above linking rules are discussed in detail [1]. For $\beta \approx 3$ the results of numerical simulations within this model (see Ref. [1]) agree satisfactorily with the empirical data on the real Web [4]. We would like to stress that the property of rewiring while the graph grows, which is enabled by $C(n, k, t) > 0$ in Eq. (1), yields qualitatively new scaling features, compared with the graphs with frozen links ($\beta \equiv 0$). For instance, one of the important consequences of rewiring is the appearance of the scale-free structure of the out-degree distribution. The entire class of graphs generated with the above rules for varying β in the range $0 < \beta < \infty$ was studied recently [7].

Here we concentrate to the growth phase of the Web graph. We fix $\beta = 3$ and chose $M = 4$. In principle, share between added and rewired links is statistical, thus the universal degree-distributions are independent on M in the scaling region [10,1]. However, some local properties can depend on the actual growth rate of the number of links. To show these dependences is another goal of this work.

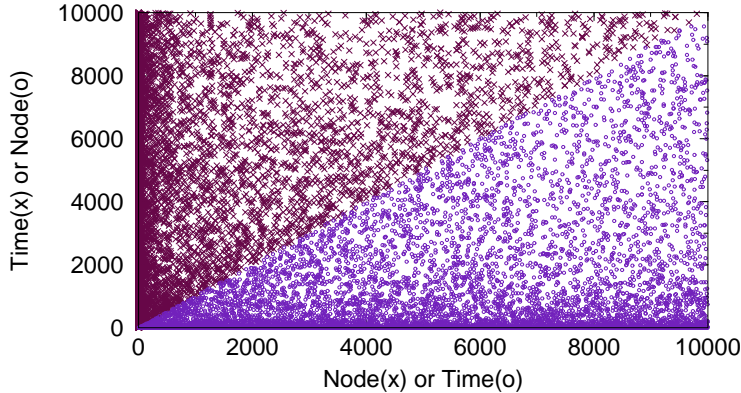


FIG. 1. Temporal patterns of activity: in-linking (crosses) and out-linking (circles) for $M = 4$ and network size $N = t = 10^4$.

In Fig. 1 we show the activity pattern of each node during the growth time $t = 1 \cdots N$, with $N = 10^4$ nodes-steps. A node n can be linked only in the time after its appearance $t \geq n$. Value on the vertical axis of a point in the plain (k, t) represents the moment of time t when the node k got an in-link (upper-left part), or a node n that created an out-link at time t (lower-right part). Quantitative characterization of these patterns can be done in two ways.

First, we notice that the time intervals between two consecutive linking activities *at a given node* are irregular, resembling a fractal set. For this reason we measure the distribution of successive time intervals Δt for in-linking and out-linking separately. The results are given in Fig. 2 (left), where the algebraic decay of the distributions $P_\kappa(\Delta t) \sim (\Delta t)^{-\mu_\kappa}$, confirms the fractal character of these sets. The slopes of the two curves are $\mu_{out} = 0.82 \pm 0.01$ and $\mu_{in} = 0.87 \pm 0.01$. Second, fixing a node s we watch how the number of links accumulates at that node with time. Averaging over a large ensembles of networks, we find that for $t \gg s$

$$\langle q_\kappa(s, t) \rangle \sim t^{\gamma_\kappa} , \quad (3)$$

where $\kappa \equiv \text{“out” or “in”}$, as shown in Fig. 2 (right), where the scaling exponents are $\gamma_{out} = 0.66 \pm 0.03$ and $\gamma_{in} = 0.87 \pm 0.03$.

III. LOCAL CONNECTIVITY AND EMERGENT DEGREE DISTRIBUTIONS

The average connectivity at a node increases as a power of evolution time for times $t \gg s$, which is compatible with scaling behavior of the local probability distribution $\rho_\kappa(q, s, t)$ that the node s collected q links in time up to the step t . It was shown [10] analytically that preference linking with the probability $p_{in}(k, t)$ given in Eq. (2) leads to the power-law behavior of the $\rho_{in}(q, s, t)$ both in q and t/s arguments. Our results in Fig. 2 suggest that, due to rewiring with the probability $C(n, k, t)$ in Eq. (1), the distribution $\rho_{out}(q, s, t)$ also exhibits scaling behavior but with different exponent ($\gamma_{out} \neq \gamma_{in}$). The emergent degree distributions $P_\kappa(q)$ are defined as

$$P_\kappa(q) = \lim_{t \rightarrow \infty} \sum_{s < t} \rho_\kappa(q, s, t) \sim q^{-\tau_\kappa}, \quad (4)$$

which we extend to both out- and in-links. In addition, the exact scaling relation that applies to in-link distributions [10] can be easily extended to out-links [11], i.e.,

$$\tau_\kappa = 1/\gamma_\kappa + 1. \quad (5)$$

Here we assume that the general scaling form applies both for in- and out-links

$$\rho_\kappa(q, s, t) \sim (s/t)^\mu f(q^x (s/t)^\Delta), \quad (6)$$

with conserved number of links of both kind, i.e., $\sum_q \rho_\kappa(q, s, t) = 1$ where $\kappa = \text{“in”}, \text{“out”}$. Then together with Eqs. (3)-(4) we find $\mu = \Delta/x = \gamma$ and $\tau = (1 + \gamma)x/\Delta$, leading to Eq. (5). The measured distributions of emergent node ranks $P_{in}(q)$ and $P_{out}(q)$ after $N = 10^5$ added nodes are shown in Fig. 3. The slopes $\tau_{out} - 1 = 1.70 \pm 0.03$ and $\tau_{in} - 1 = 1.26 \pm 0.02$ obey the scaling relation (5) with the respective values for γ_{out} and γ_{in} taken from average connectivity in Fig. 2.

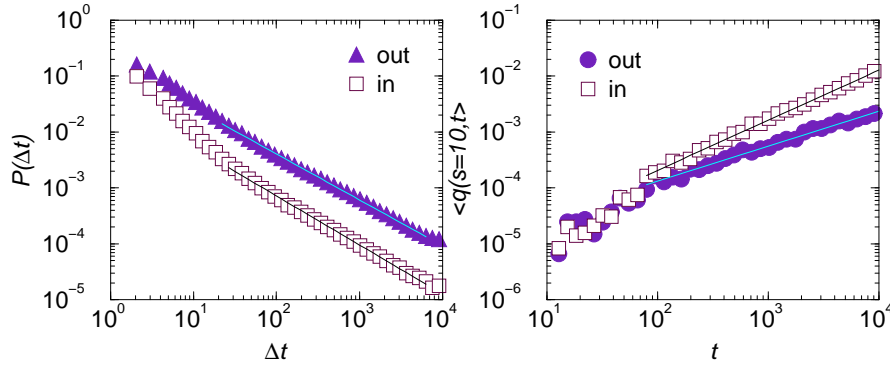


FIG. 2. Left panel: Distributions of return-time for out- and in-linking. Right panel: Average connectivity at node $s = 10$ vs. evolution time t . $N = 10^4$, $M = 4$, data log-binned, bin ratio 1.2.

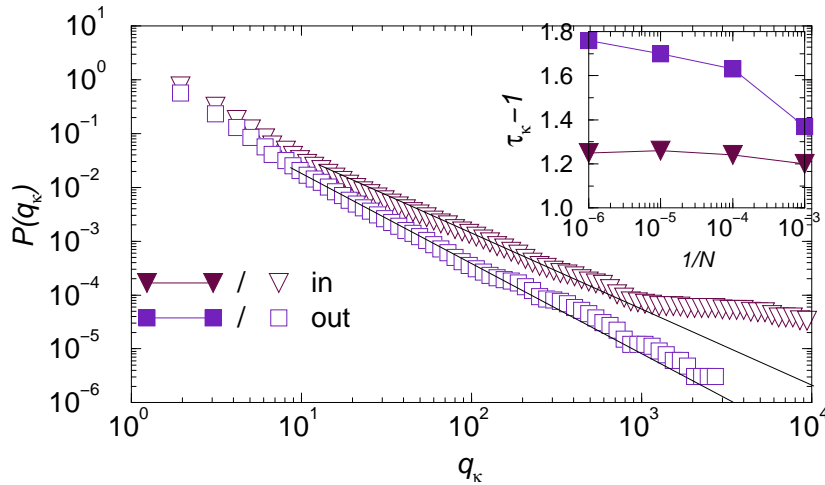


FIG. 3. Cumulative distributions of node degrees for out- and in-links after $t = N = 10^5$ evolution steps. $M = 1$, log-binning ratio 1.1. Inset: Corresponding scaling exponents vs. $1/N$.

IV. WALKER ON THE WEB: LOCAL STRUCTURES

We have demonstrated that evolution of local organization both of out- and in-links at individual nodes is responsible for the global scaling of the emergent node distributions for large times and number of nodes. Here we would like to show that dynamic processes on grown network (i.e., at different time scale) which use the information on these local properties also obey certain scaling laws. Such processes on the Web are different kinds of random walks related, for example, to search algorithms. We define two types of random walks [6,7]: The adaptive random walk (ARW) that selects the target node k_ℓ with the weight which is proportional to in-linking probability of the visited node, and a naive random walk (NRW) selecting one of the out links with equal probability. The corresponding weights are

$$w_{ARW}(n, k_\ell) = p_{in}(k_\ell) / \sum_{\ell=1}^{q_{out}(n)} p_{in}(n, k_\ell) , \quad w_{NRW}(n, k_\ell) = 1/q_{out}(n) . \quad (7)$$

In Fig. 4 we show the distributions of distances in hierarchy levels Δq_κ inside the clusters of *connected nodes* which are visited in cumulative time by an ensemble of walkers. As the Fig. 4 shows, these local clusters are organized scale-free structures with $W(\Delta q_\kappa) \sim (\Delta q_\kappa)^{-\delta_\kappa}$. Notably the scaling exponents $\delta_{out} \approx \delta_{in}$ expressing the correlations due to normalization of weights in Eq. (7). For the case of ARW the exponents are close to τ_{in} of the global structure of in-links, whereas for the clusters scanned by NRW they are reduced by approximately unity, i.e., 2.07 ± 0.04 and 1.10 ± 0.03 for ARW and NRW, respectively.

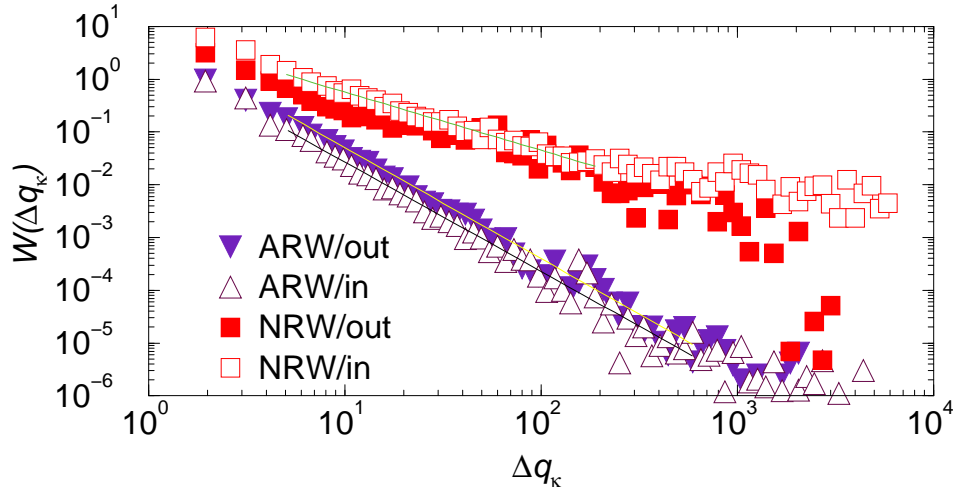


FIG. 4. Time-integrated distributions of distances of out- and in-degree Δq , made by adaptive (ARW) and naive (NRW) walkers on the Web graph. $N = 10^4$, $M = 4$, log-binning ratio 1.1 .

V. CONCLUSIONS

The probabilistic character of linking rules for $\alpha < 1$ with self-consistently varying linking probabilities leads to power-law behavior of local and global (emergent) link structures, both for out- and in-links. This numerical results (see also references [1,6,7]) are in agreement with the analytical results obtained by rate equation approach in the same model [11]. We have demonstrated here that the basis of these scaling laws lies in the occurrence of dynamic fractals and hence the algebraic growth of the local connectivity in time, from which then the hierarchical global structure emerges at large times.

In addition, such local connectivity affects random-walk processes on grown networks. The connected clusters (subgraphs) scanned by the random-walk ensembles also exhibit scaling behavior of distances in node degrees. The scaling properties of these subgraphs on the Web strictly depend on the applied random walk strategy, i.e., in the degree of information about local connectivity that the walkers use. Moreover, the scaling exponents of the distributions of distances in these subgraphs on the Web decrease with increasing rate M , in contrast to the global structure of the graph, which is universal for large network size N .

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* E-mail: Bosiljka.Tadic@ijs.si

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